Indian Statistical Institute, Bangalore
B.Math (Hons.) I Year, Second Semester
Semestral Examination
Real Analysis II, Back Paper

Time: 3 hours | July 18, 2011 | Instructor: C.R.E.Raja
Maximum marks: 50

Each question is worth 10 Marks

1. Let E be a subset of a metric space X. Prove that (a) $\overline{E} = E' \cup E$, (b) $\overline{E}$ is a closed set and (c) $\overline{E}$ is the intersection of all closed sets in X containing E.

2. Let (X, d) be a discrete metric space.

   (a) Suppose (a_n) is a convergent sequence in X. Prove that there is a $N \geq 1$ such that $a_n = a_m$ for all $n, m \geq N$

   (b) Prove that a subset K of X is compact if and only if K is finite.

   (c) Prove that a non-empty subset E of X is connected if and only if E is singleton.

3. (a) If $f, g \in \mathcal{R}[a, b]$, then prove that $f + g \in \mathcal{R}[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

   (b) If $f\colon [a, b] \to \mathbb{R}$ is a monotonic function, then prove that $f \in \mathcal{R}[a, b]$.

4. (a) Let $f, g \in \mathcal{R}[a, b]$ and $g(x) \geq 0$ for all $x \in [a, b]$. Prove that there is a $s \in \mathbb{R}$ such that $\int_a^b fg = s \int_a^b g$ and $\inf_{x\in[a,b]} f(x) \leq s \leq \sup_{x\in[a,b]} f(x)$.

   (b) Let $f\colon \mathbb{R}^k \to \mathbb{R}^k$ be a differentiable function. Prove that f is continuous.

5. (a) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function and $x, y \in \mathbb{R}^n$. Prove that there is a z on the line segment joining x and y such that $f(y) - f(x) = f'(z)(y - x)$.

   (b) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $u \in \mathbb{R}^n$ be such that $||u|| = 1$. Prove that $\lim_{t\to 0} \frac{f(x+tu)-f(x)}{t}$ exists and equal to $\sum_{i=1}^n D_i f(x) u_i$ where u_i is the i-th co-ordinate of u.